Nonquasilinear evolution of particle velocity in incoherent waves with random amplitudes

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Abstract

The one-dimensional motion of N particles in the field of many incoherent waves is revisited numerically. When the wave complex amplitudes are independent, with a gaussian distribution, the quasilinear approximation is found to always overestimate transport and to become accurate in the limit of infinite resonance overlap.

Key words: weak plasma turbulence, stochastic acceleration, hamiltonian chaos, transport
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The motion of a particle in the field of many waves is a fundamental process in classical physics [7,8,20], usually discussed in the first chapters of plasma physics textbooks. For modeling purposes, this process can be described by the hamiltonian model

$$H = \frac{v^2}{2} + \sum_{m=1}^{M} A_m \cos(k_m x - \omega_m t - \varphi_m) \tag{0.1}$$

where the particle has position x and momentum v (normalizing mass to 1 and writing $A_m = qE_m/k_m$). The wave field has $M \gg 1$ waves, with a smooth dispersion relation associating a wavenumber k_m, a pulsation ω_m and a phase velocity $v_m = \omega_m/k_m$ to each wave – usually determined by fixed properties of the environment, such as the geometry of the domain where waves propagate (then the k_m are discrete). The actual spectrum of the wavefield, given by the complex amplitudes $A_m \mathrm{e}^{\mathrm{i}\varphi_m}$, is more easily tuned by the experimenter or affected by simple changes in the environment, e.g. excited antennas.

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The dynamical system approach to this problem would be to prescribe a single choice for each wave complex amplitude. However it would be quite exceptional to control all waves (though this is e.g. the assumption underlying the standard map with $A_m = A_0$, $\varphi_m = \varphi_0$ for all waves – see [3] for a discussion).

Because the waves have different frequencies and velocities, it is generally unrealistic to assume their phases to be correlated. Their intensities are more easily observed, but both in nature and in the laboratory the accumulation of statistics on waves often involves only their average power spectra, not the detailed intensity data for each measurement run. Besides, if the waves are excited by a noisy source, as occurs in the self-consistent dynamics of particles and waves [9,12], one cannot expect a systematic reproduction of individual wave data among several experiments (this is the very meaning of a "noisy source").

We thus assume here that these complex amplitudes are random data, and investigate the statistics of the particle motion in the resulting time-dependent random field. This dynamics is a "stochastic acceleration problem" for a "passive particle" in weak plasma turbulence [6,19,21], and its understanding is a prerequisite to a proper analysis of the case where the particle motion feeds back on the wave evolution [12].

This hamiltonian generates equations of motion

$$\dot{x} = v\,, \tag{0.2}$$
$$\dot{v} = \sum_m k_m A_m \sin(k_m x - \omega_m t - \varphi_m)\,. \tag{0.3}$$

A simple case for the dispersion relation is

$$k_m = k_0,\ \ \omega_m = 2\pi(m - M/2)/T, \tag{0.4}$$

for some k_0, T. Then, in the limit $M \to \infty$, the equations of motion yield for $A_m \mathrm{e}^{\mathrm{i}\varphi_m} = A_0$ (with real A_0) the well-known standard map [3]. The case where phases φ_m are independent random variables uniformly distributed on the circle $[0, 2\pi]$ while $A_m = A_0$ is given was investigated notably by Cary, Escande, Verga and Bénisti [1,4,12] and occurs in the context of the random phase approximation.

An important observation [5,14] on the motion of a particle in the field (0.3) is locality in velocity : the evolution of the particle when it has velocity v depends only weakly on the waves with a Doppler-shifted frequency $\omega_m - k_m v$ much larger than their trapping oscillation frequency $2k_m\sqrt{A_m}$. In particular,

for a two-wave system the resonance overlap parameter

$$s_{1,2} = \frac{2\sqrt{A_1} + 2\sqrt{A_2}}{|\omega_2/k_2 - \omega_1/k_1|} \tag{0.5}$$

becomes unity when there exists a velocity u for which $\omega_2/k_2 - u = 2\sqrt{A_2}$ and $u - \omega_1/k_1 = 2\sqrt{A_1}$ (with $k_1 > 0$, $k_2 > 0$, $\omega_2 > \omega_1$). For many waves with overlap parameters $s \gg 1$, the relevant phase velocity range for waves influencing the particle is a "resonance box", with a width scaling as $A^{2/3}$ [1,2].

To the extent that the phases and amplitudes of the waves are independent random variables, it is tempting to approximate the acceleration (0.3) by a white noise, with amplitude $\sigma = \sqrt{\mathbb{E}\,(k_m^2 A_m^2)} = k_m \sqrt{\mathbb{E}\,A_m^2}$, where the relevant mode m is the one with the current particle velocity ($\mathbb{E}$ denotes the mathematical expectation, or ensemble average with respect to wave amplitudes and phases). [2]

This is the core of quasilinear theory [10,17,18,22]. Mathematically, one then interprets (0.2)-(0.3) as a stochastic differential equation, after the necessary reformulation to make sense of divergent series ; this is correct in the case $m \in \mathbb{Z}$ for dispersion relation (0.4) with gaussian independent complex amplitudes such that $\mathbb{E}\,A_m^2 = k_0^{-2}\sigma^2$. The particle velocity then has a brownian evolution, so that for $0 < t \leq t' < T$

$$\mathbb{E}\,(v_t - v_0)(v_{t'} - v_0) = 2 D_{\mathrm{QL}} \min(t, t') \tag{0.6}$$

with the quasilinear diffusion coefficient $D_{\mathrm{QL}} = \sigma^2 T/4$. In particular,

$$\mathbb{E}\,\Delta v_t^2 = \frac{\sigma^2 T}{2}\, t \tag{0.7}$$

for $\Delta v_t = v_t - v_0$. However, the particle evolution for $t > T$ may show a strong correlation to its motion for $0 \leq t \leq T$ because the waves are periodic in time [11,13], and the dispersion relation (0.4) may generate a strong spatial correlation – although it just accounts for the fact that in the limit of a dense spectrum (with a smooth dispersion relation) the relevant waves acting on a particle at any time also have almost the same wavelength. [3]

[2] Phases do not appear in σ (nor in s) because the relative phase of two waves $\varphi_m + \omega_m t - \varphi_n - \omega_n t$ varies uniformly over time (hence $\varphi_m - \varphi_n$ can be absorbed in the choice of the time origin).

[3] There is a large body of literature on the case of incoherent waves with no dispersion relation. Then the sum $\sum_m$ becomes a double sum $\sum_{m,n}$ and one varies wavenumbers k_n independently from pulsations ω_m. This space-time stochastic environment is more noisy that our model and may also be considered to motivate a quasilinear approximation.

In this note we compare numerical properties of the velocity evolution for fixed amplitude with the case of random wave amplitudes : let $A_m e^{i\varphi_m} = C_m + iS_m$. In the latter case, the amplitudes are drawn independently in such a way that their squares have exponential distributions, with equal expectation $\mathbb{E}\, A_m^2 = k_0^{-2}\sigma^2$. Along with uniform distribution of the phases, this is equivalent to gaussian distribution of the complex amplitudes. Indeed, the probability density f_φ for φ is uniform, and the exponential law for A^2 yields for A the density at $a > 0$

$$f_A(a) = \frac{\mathbb{P}(a \le A < a + \,\mathrm{d}a)}{\mathrm{d}a} = \frac{\mathbb{P}(a^2 \le A^2 < a^2 + 2a\,\mathrm{d}a)}{\mathrm{d}a} = \mathrm{e}^{-a^2/\mathbb{E}\,A^2}\frac{2a}{\mathbb{E}\,A^2}\,. \tag{0.8}$$

The probability density for (C, S) is thus (noting that $a^2 = c^2 + s^2$ and $\mathrm{d}c\,\mathrm{d}s = a\,\mathrm{d}a\,\mathrm{d}\varphi$)

$$f_{C,S}(c, s) = \frac{f_A(a) f_\varphi(\varphi)\,\mathrm{d}a\,\mathrm{d}\varphi}{\mathrm{d}c\,\mathrm{d}s} = \frac{2a\mathrm{e}^{-a^2/\mathbb{E}\,A^2}}{2\pi\,\mathbb{E}\,A^2}\frac{\mathrm{d}a\,\mathrm{d}\varphi}{\mathrm{d}c\,\mathrm{d}s} = \frac{\mathrm{e}^{-(c^2+s^2)/\mathbb{E}\,A^2}}{\pi\,\mathbb{E}\,A^2} \tag{0.9}$$

which is the density for a gaussian vector (C, S) with zero expectation and covariance matrix $\begin{pmatrix} \mathbb{E}\,A^2/2 & 0 \\ 0 & \mathbb{E}\,A^2/2 \end{pmatrix}$.

For the simulations, we let $k_0 = 1$ and $T = 2\pi$ so that $\sigma = s^2/16$. Given the value of σ, or equivalently of s, we draw at random $N_{\mathrm{pha}} = 400$ sets of wave data. For each set of wave data, we follow the evolution of $N_{\mathrm{orb}} = 50$ particles, released initially at random points (x, p) in the strip $0 \le x \le 2\pi$, $-0.5 \le p \le 0.5$. Trajectories are computed with a reversible symplectic integrator up to a time, large enough for the motion to exhibit possible departure from quasilinear statistics, but short enough to ensure that the particles remain far from the boundaries of the wave spectrum, which have velocity $\pm M/2$ (where we take M up to 800) : for $s = 13$, the boundaries are more than 3 standard deviations (which is about 120 as observed in Fig. 1) away from the initial particle velocity. Some trajectories are computed backward from their final point to control numerical accuracy : the accumulated error is about 10^{-5} for $s = 3.5$ for the runs of Fig. 1 and 3, though it deteriorates rapidly with increasing s. Our calculations for the equal A case reproduce the findings of Refs [1,4].

Statistical averages discussed below (denoted by $\langle\cdot\rangle$) are performed over all particle and wave data for the same s value. We focus on the second and fourth moments of particle velocity as functions of time. To the numerical average $g_2(t) = \langle \Delta v^2 \rangle$ we fit a linear approximation $D_0 + 2D_{\mathrm{eff}}t$, which defines an effective diffusion coefficient. Such a plot is displayed in Fig. 1. The quasilinear prediction $2D_{\mathrm{QL}}t$ is a straight line. The numerical average g_2 is plotted, along with two lines estimating one standard deviation on either side, viz. $g_2 \pm \langle \Delta v^4 -$

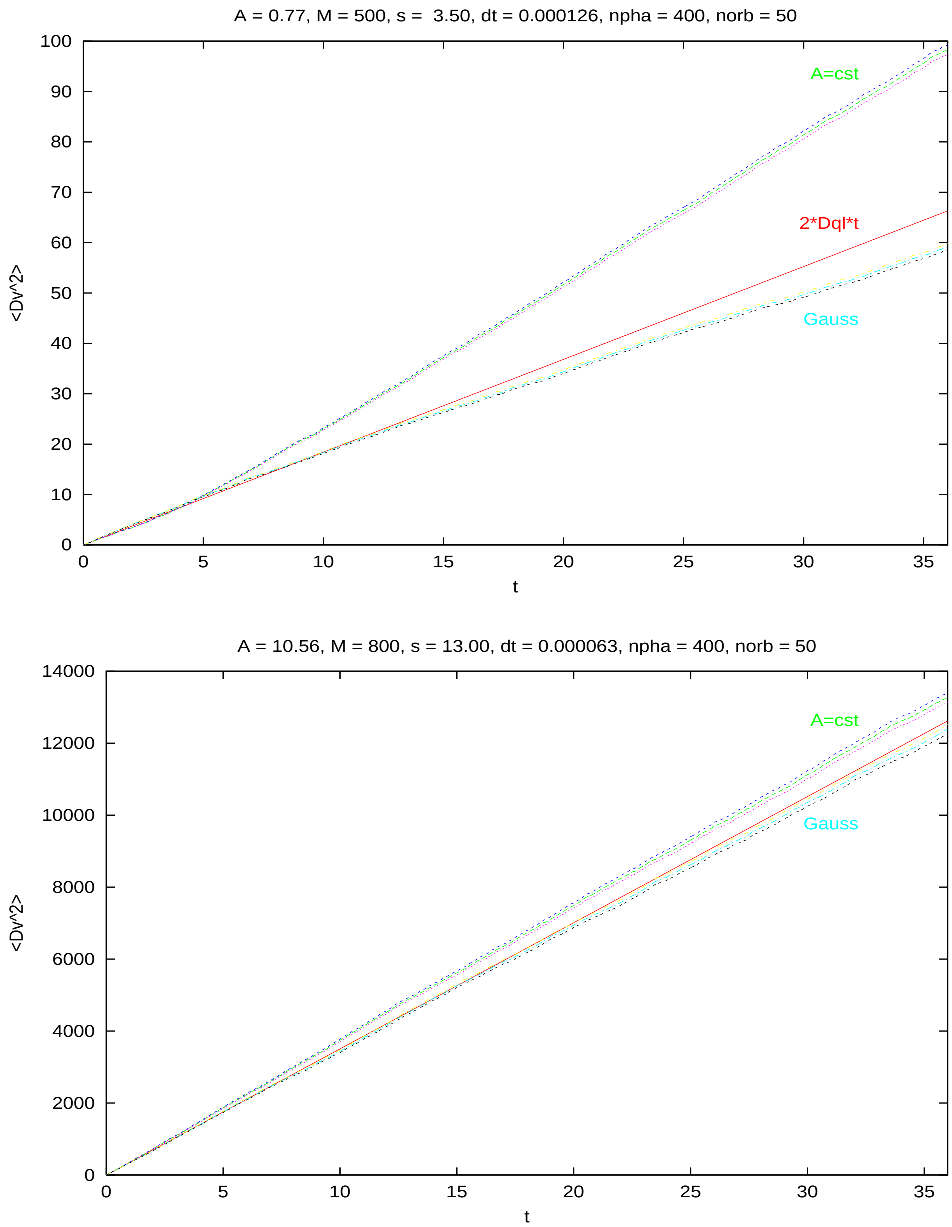

Fig. 1. Dependence of $\langle \Delta v^2 \rangle$ on time for random amplitudes (Gauss) and for equal amplitudes (A=cst).

$g_2^2\rangle^{1/2}$. The narrowness of the channel so constructed indicates the statistical accuracy of the plotted function g_2.

Specifically, Fig. 1 displays our results for $s = 3.5$ and for $s = 13$. The non-overlap regime ($s \lesssim 1$) allows no large scale transport in velocity, because the particle phase space contains invariant Kolmogorov-Arnol'd-Moser tori. Indeed the case $s = 0$ is integrable (it corresponds to free particle motion). For increasing overlap parameter value, one expects the transport to become increasingly chaotic, and indeed the particle dynamics typically has a Liapunov

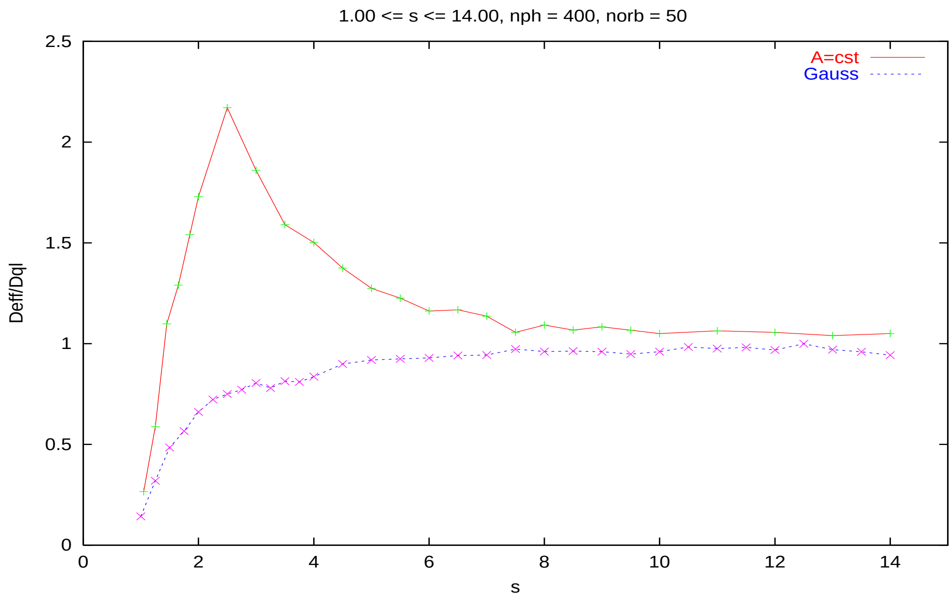

Fig. 2. Ratio $D_{\text{eff}}/D_{\text{QL}}$ as function of overlap parameter s for random amplitudes (Gauss) and for equal amplitudes (A=cst). The lines are guides to the eye.

exponent scaling like $A^{1/3} \sim s^{2/3}$ for large s [12]. One may then expect the effective diffusion coefficient to increase towards the quasilinear estimate, D_{QL}, which corresponds to a pure stochastic particle behaviour. However it has been found [1,4,12] that for the equal amplitude, independent phases wave data, the particle velocity undergoes an enhanced transport, with an effective diffusion coefficient up to 2.5 times the quasilinear estimate, and the latter becomes accurate only in the large s limit. It must be noted that this enhanced transport shows up only after time $T = 2\pi$, i.e. only on a time scale for which the time periodicity due to the discrete nature of the wave spectrum significantly affects the particle motion – actually the motion for short times is essentially quasilinear thanks to the wave stochasticity [12,16]. We also checked that the force correlation function on the particle is essentially zero, except for times near integer multiples of the period T.

Our simulations show that the enhanced transport is not observed for independent wave amplitudes. In the latter case, the velocity variance grows with time at a rate never exceeding the quasilinear value. As shown in Fig. 2 this observation holds for the whole range of overlap parameter values sampled, in agreement with the naive prediction that "increasing chaos" should make dynamics more similar to "pure noise".

The noisiness of the particle evolution is also assessed using a higher-moment test. If the velocity distribution at time t were gaussian, the ratio $\langle \Delta v^4 \rangle / (3 g_2^2)$ would equal 1. Fig. 3 displays numerical evidence that for the equal wave amplitude case this ratio remains rather below 1 (the velocity distribution has smaller tails than the gaussian with equal variance), whereas for the gaussian

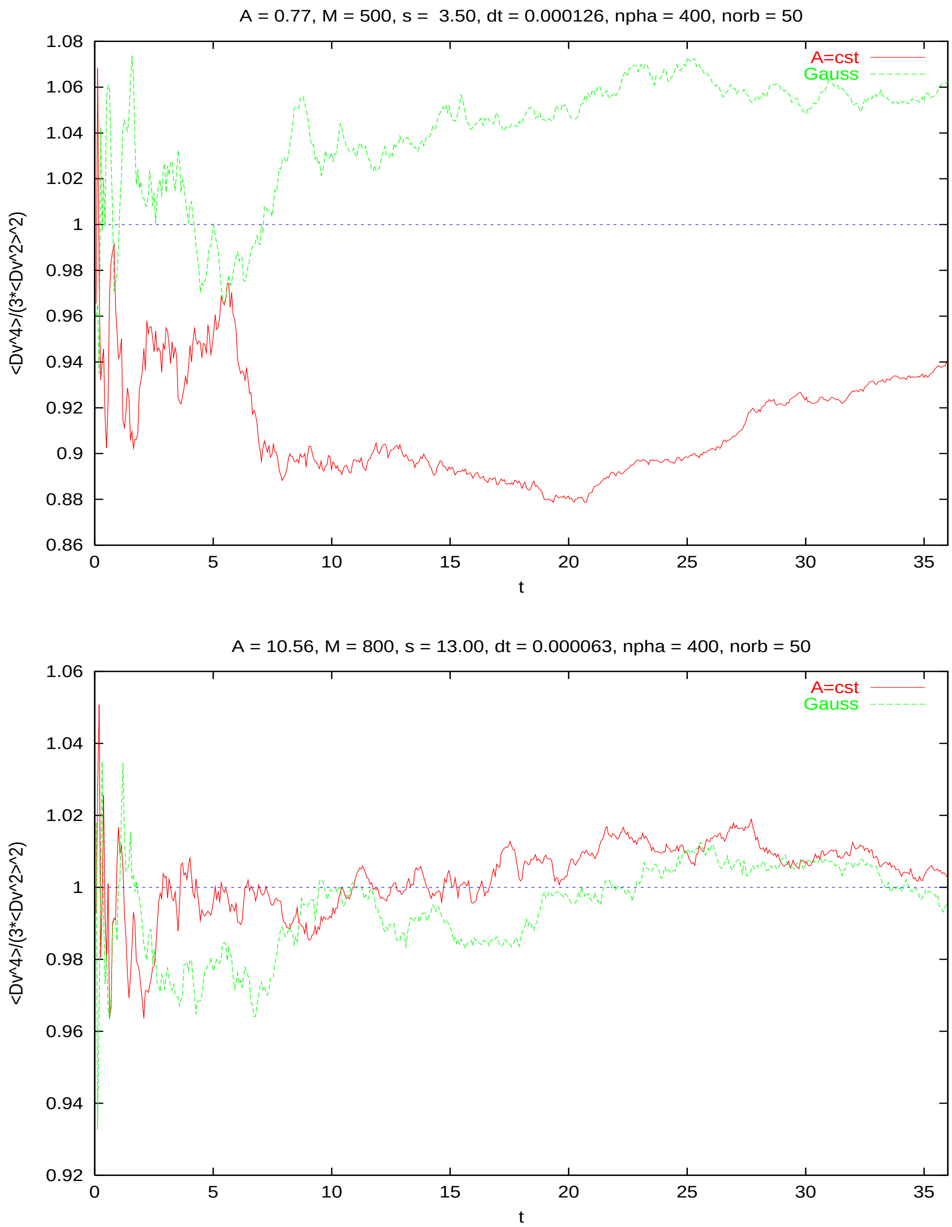

Fig. 3. Ratio $\langle \Delta v^4 \rangle / (3 \langle \Delta v^2 \rangle^2)$ versus time for random amplitudes (Gauss) and for equal amplitudes (A=cst).

wave data it is somewhat above 1 (the distribution has stronger tails than the gaussian with same variance).

For both wave data types the limiting behaviour for $s \to \infty$ is quasilinear, in agreement with theoretical [1,12,15] and mathematical [11,13] arguments.

These results confirm that random phases only are not sufficient to substantiate the quasilinear approximation for the stochastic acceleration problem. A gaussian wave spectrum may seem closer to the ideal view of white noise for

one period T, but over longer times correlations also build up, driving transport away from the quasilinear approximation, especially when the overlap parameter has moderate typical values ($1 \lesssim s \lesssim 5$).

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